

Phase Synchronization

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1 Introduction

For any mode or fluctuation, we always have

$$\phi_k = |\phi_k(x, t)| e^{iS(x, t)} \quad (1)$$

where $S(x, t)$ is phase.

If a mode amplitude satisfies

$$|\phi_k(x, t)| \propto e^{\gamma t} \quad (2)$$

and $\gamma \rightarrow 0$, we call it the marginal state. Near the marginal state, we have

$$\partial_t \phi_k \sim \partial_t e^{iS(x, t)} \sim \partial_t S(x, t) \quad (3)$$

This means near the marginal state, the evolution of the mode is essentially determined by its phase. That's why phase dynamics matters!

References are listed in the end of this note.

This lecture will cover the following two topics:

- A nonlinear Limit-Cycle-Oscillator (LCO) – Stuart-Landau Equation
- Collective behavior of many LCOs – Kuramoto Model

2 A Self Interacted LCO: Autonomous System

In this section, we study an autonomous system that satisfy the following first order differential equation:

$$\frac{d}{dt}\mathbf{X} = F(\mathbf{X}; \mu) \quad (4)$$

where $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is a vector, μ means the scalar parameters in this system. Eq. (4) is a very general form, many specific systems can be written in the above form. Examples of μ : it can be temperature, density or gradients of them in plasma physics.

For some range of μ , the system may stay stable; when μ exceeds some critical value μ_c , the system may give a periodic motion solution. The critical point where a system's stability switches and a periodic solution arises is called the **Hopf bifurcation**.

- $\mu < \mu_c$: The system stays in a fixed or stationary state.
- $\mu > \mu_c$: The oscillator evolves into LCO state.

An example: simple harmonic oscillator:

$$\frac{d}{dt}x = v \quad (5)$$

$$\frac{d}{dt}v = -kx \quad (6)$$

Here X_1 is x and X_2 is v , the scalar parameter μ is k . The critical value is $k_c = 0$: when $k = 0$ the system is stationary and when $k > 0$ the solution is periodic motion.

Let $\mathbf{X}_0(\mu)$ denote a steady solution of Eq. (4), i.e. $F(\mathbf{X}_0(\mu); \mu) = 0$. Note that there is always a stationary solution, the question is whether this solution is stable. We express Eq. (4) in terms of the deviation $\mathbf{u} = \mathbf{X} - \mathbf{X}_0$ in a Taylor series:

$$\frac{d}{dt}\mathbf{u} = L\mathbf{u} + M\mathbf{u}\mathbf{u} + N\mathbf{u}\mathbf{u}\mathbf{u} + \dots \quad (7)$$

where

$$(L\mathbf{u})_i = \sum_j \frac{\partial F_i(\mathbf{X}_0)}{\partial X_{0j}} u_j \quad (8)$$

$$(M\mathbf{u}\mathbf{u})_i = \sum_{jk} \frac{1}{2!} \frac{\partial^2 F_i(\mathbf{X}_0)}{\partial X_{0j} \partial X_{0k}} u_j u_k \quad (9)$$

$$(N\mathbf{u}\mathbf{u}\mathbf{u})_i = \sum_{jkl} \frac{1}{3!} \frac{\partial^3 F_i(\mathbf{X}_0)}{\partial X_{0j} \partial X_{0k} \partial X_{0l}} u_j u_k u_l \quad (10)$$

2.1 Order by Order Expansion

Now assume $\mu_c = 0$ for simplicity. Near criticality, suppose that μ is small, and do a reduced perturbation expansion:

$$L(\mu) = L_0 + \mu L_1 + \mu^2 L_2 + \dots \quad (11)$$

It is convenient to define a small positive parameter ε by $\varepsilon^2 \chi = \mu$, where $\chi = \text{sgn}(\mu)$; ε is considered to be a measure of the amplitude to lowest order, so that one may assume the expansion:

$$u = \varepsilon u_1 + \varepsilon u_2 + \dots \quad (12)$$

Note that μ is 'equilibrium profile' change, so it is smaller than the fluctuation amplitude change, that's why μ is assumed to be of order ε^2 . Thus Eq. (11) now becomes

$$L = L_0 + \varepsilon^2 \chi^2 L_1 + \varepsilon^4 L_2 + \dots \quad (13)$$

Similarly,

$$M = M_0 + \varepsilon^2 \chi^2 M_1 + \varepsilon^4 M_2 + \dots \quad (14)$$

$$N = N_0 + \varepsilon^2 \chi^2 N_1 + \varepsilon^4 N_2 + \dots \quad (15)$$

All $u_\nu(t)$ ($\nu = 1, 2, \dots$) include fast and slow variations! We need to expand the time coordinate, too. The slow variation is associated with profile change (related to μ), so it is plausible to assume

$$\frac{d}{dt} \sim \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial \tau} \sim \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau} \quad (16)$$

Now we can expand L, M, N, u , and $\frac{d}{dt}$, so we expand Eq. (7) order by order:

$$\varepsilon^1 : \quad \left(\frac{\partial}{\partial t} - L_0\right)u_1 = 0 \quad (17)$$

$$\varepsilon^2 : \quad \left(\frac{\partial}{\partial t} - L_0\right)u_2 = M_0 u_1 u_1 \quad (18)$$

$$\varepsilon^3 : \quad \left(\frac{\partial}{\partial t} - L_0\right)u_3 = -\left(\frac{\partial}{\partial \tau} - \chi L_1\right)u_1 + N_0 u_1 u_1 u_1 + 2M_0 u_1 u_2 \quad (19)$$

$$(20)$$

Generally,

$$\left(\frac{\partial}{\partial t} - L_0\right)u_\nu = B_\nu \quad (21)$$

where

$$B_1 = 0 \quad (22)$$

$$B_2 = M_0 u_1 u_1 \quad (23)$$

$$B_3 = -\left(\frac{\partial}{\partial \tau} - \chi L_1\right)u_1 + N_0 u_1 u_1 u_1 + 2M_0 u_1 u_2 \quad (24)$$

$$(25)$$

2.2 The solvability condition

L_0 is the fast time scale. In order to remove the fast time scale, here we introduce the left and right eigenvector:

$$L_0 U = \lambda U \quad (26)$$

$$U^\dagger L_0 = \lambda U^\dagger \quad (27)$$

The first two eigenvalues can be expressed as $\lambda_0 = i\omega_0 = U^* L_0 U$, and $\lambda_1 = \sigma_1 + i\omega_1 = U^* L_1 U$.

Multiply U^\dagger by Eq. (21), and we obtain the solvability condition:

$$\int_0^{2\pi} U^\dagger \left(\frac{\partial}{\partial t} - L_0\right)u_\nu e^{-i\omega_0 t} dt = \int_0^{2\pi} U^\dagger B_\nu e^{-i\omega_0 t} dt = 0 \quad (28)$$

B_ν is 2π -periodic in $\omega_0 t$, so we can express it in this form:

$$B_\nu(t, \tau) = \sum_{l=-\infty}^{\infty} B_\nu^{(l)}(\tau) e^{il\omega_0 t} \quad (29)$$

The solvability condition then becomes:

$$\boxed{U^\dagger B_\nu^{(l)}(\tau) = 0} \quad (30)$$

This condition is crucial.

2.3 Stuart-Landau Equation

For $\nu = 1$, the solution for Eq. (21) has this form:

$$u_1 = W(\tau)Ue^{i\omega_0 t} + c.c. \quad (31)$$

where c.c. means complex conjugate, and $W(\tau)$ is some complex amplitude yet to be solved. In order to obtain an equation for W , we need Eq. (30) for $\nu = 2$ and $\nu = 3$.

For $\nu = 2$, the Eq. (30) is trivially satisfied because $U^\dagger U = 0$.

For $\nu = 3$, express u_2 in terms of u_1 (or W). u_2 contains only zeros and second harmonics, so we assume u_2 has this form:

$$u_2 = V_+ W^2 e^{2i\omega_0 t} + V_- \bar{W}^2 e^{2i\omega_0 t} + V_0 |W|^2 + v_0 u_1 \quad (32)$$

plug it into Eq. (21) for $\nu = 2$, the coefficients are obtained:

$$V_+ = \bar{V}_- = -(L_0 - 2i\omega_0)^{-1} M_0 U U \quad (33)$$

$$V_0 = -2L_0^{-1} M_0 U \bar{U} \quad (34)$$

Now substitute all the above expressions into B_3 , the solvability equation for $\nu = 3$ becomes:

$$\boxed{\frac{\partial W}{\partial \tau} = \chi \lambda_1 W - g |W|^2 W} \quad (35)$$

where g is a complex number given by

$$g = g' + ig'' = -2U^* M_0 U V_0 - 2U^* M_0 \bar{U} V_+ - 3U^* N_0 U U \bar{U} \quad (36)$$

Eq. (35) is the **Stuart-Landau Equation**.

Write W into real and imaginary parts: $W = Re^{i\Theta}$, then Eq. (35) becomes the following two equations:

$$\frac{dR}{d\tau} = \chi \sigma_1 R - g' R^3 \quad (37)$$

$$\frac{d\Theta}{d\tau} = \chi \omega_1 - g'' R^2 \quad (38)$$

The trivial solution is:

$$R = 0 \tag{39}$$

$$\Theta = \chi\omega_1\tau + \text{const} \tag{40}$$

The nontrivial solution is:

$$R = R_s \tag{41}$$

$$\Theta = \bar{\omega}\tau + \text{const} \tag{42}$$

where $R_s = \sqrt{\sigma_1/|g'|}$, and $\bar{\omega} = \chi(\omega_1 - g''R_s^2)$ is the nonlinear frequency. Marginal case requires $\frac{dR}{d\tau} = 0 \Rightarrow \chi\sigma_1 = g'R^2$. Because $R^2 \geq 0$ is always true, the nontrivial solution appear only in the supercritical region ($\chi > 0$) for $g' > 0$, and in the subcritical region ($\chi < 0$) for $g' < 0$.

This bifurcating solution shows a perfectly smooth circular motion in the complex W plane. The original vector variable \mathbf{X} is approximately given by:

$$\mathbf{X} = \mathbf{X}_0 + \varepsilon\mathbf{u}_1 = \mathbf{X}_0 + \varepsilon(\mathbf{U}R_s e^{i(\omega_0 + \varepsilon^2\bar{\omega})t} + c.c.) \tag{43}$$

which describes a small amplitude elliptic orbital motion in a critical eigenplane.

3 Phase Dynamics of Many Oscillators

Review of the previous section: The equation for autonomous system near marginal state is

$$\frac{d}{dt}\mathbf{X} = F(\mathbf{X}; \mu) \tag{44}$$

After long algebra, we obtain the **Stuart-Landau Equation**:

$$\frac{\partial W}{\partial \tau} = \chi\lambda_1 W - g|W|^2 W \tag{45}$$

Now let's consider a system that contains many oscillators. Each oscillator oscillates in its own frequency, plus there's a weak coupling:

$$\frac{\partial W_i}{\partial \tau} = \chi\lambda_1 W_i - g|W|^2 W_i + \sum_j F(W_i, W_j) \tag{46}$$

3.1 The Order Parameter

Firstly, we consider linear coupling:

$$F(W_i, W_j) = F_{ij}W_j \quad (47)$$

For simplicity, choose $F_{ij} = K/N$. Further, assume all oscillators has the same amplitudes $|W_i| = |W|$ for all i , where $W_i = |W_i|e^{i\theta_i}$. Then the original equation becomes

$$\frac{\partial W_i}{\partial \tau} = \chi\lambda_1 W_i - g|W|^2 W_i + \frac{K|W|}{N} \sum_j e^{i\theta_j} \quad (48)$$

If θ_j distributes randomly, we have

$$\frac{1}{N} \sum_j e^{i\theta_j} = 0 \quad (49)$$

On the other hand, if θ_j concentrates at a common value: $\theta_j = \Psi$ for all j , we have

$$\frac{1}{N} \sum_j e^{i\theta_j} = e^{i\Psi} \quad (50)$$

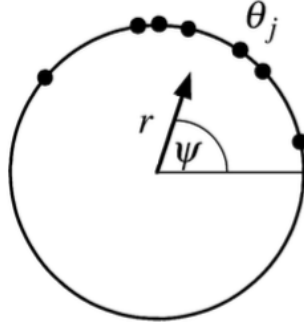


Figure 1: Geometric interpretation of the order parameter r . The phases θ_j are plotted on the unit circle. Their centroid is given by the complex number $re^{i\Psi}$, shown as an arrow.

In other words,

$$\frac{1}{N} \sum_j e^{i\theta_j} = re^{i\Psi} \quad (51)$$

with $0 \leq r \leq 1$, r acts as an order parameter. $r = 0$ means totally incoherent; $r = 1$ means totally coherent.

3.2 Kuramoto Model

From Eq. (48), we can focus on its phase evolution equation after some trivial algebra:

$$\frac{\partial \theta_i}{\partial \tau} = \omega_i + \frac{K}{N} \sum_j \sin(\theta_j - \theta_i) \quad (52)$$

where ω_i represents the frequency for i . This is the **Kuramoto model**. The major assumptions of the Kuramoto model:

- All-to-all couplings
- Equally weighted (const K)

The frequency has a distribution $g(\omega)$. $K > 0$ is attractive coupling; $K < 0$ is repulsive coupling.

There are two competing processes: one is oscillating in its eigenfrequency, the other one is entrainment by other oscillators.

Here we are mostly interested in the $K > 0$ case.

With Eq. (51), Eq. (52) can be simplified:

$$\frac{\partial \theta_i}{\partial \tau} = \omega_i + Kr \sin(\Psi - \theta_i) \quad (53)$$

Ψ is a collective phase. It has the form of $\Psi(t) = \Omega t$. It can always be removed by going into a rotating (Ω) reference frame. So without loss of generality, we simplify it by assuming $\Psi = 0$:

$$\frac{\partial \theta_i}{\partial \tau} = \omega_i - Kr \sin(\theta_i) \quad (54)$$

This is the **Adler Equation**.

3.3 Phase Lock and Phase slip

Let's analyze the Adler Equation. Note that $|\sin(\theta_j)| \leq 1$ is always true.

1. If $|\omega_i| < Kr$: $\frac{\partial \theta_i}{\partial \tau} = 0 \Rightarrow \sin \theta_i = \frac{\omega_i}{Kr}$. This is the phase lock case.
2. If $|\omega_i| > Kr$: $\frac{\partial \theta_i}{\partial \tau} \neq 0$. $\theta_i = 2 \tan^{-1}[\frac{1}{A} + \frac{\sqrt{A^2-1}}{A} \tan \frac{Kr\sqrt{A^2-1}}{2}(\tau - \tau_0)]$, where $A = \frac{\omega_i}{Kr}$. $\omega_{slip} = \frac{1}{2}\sqrt{\omega_i^2 - K^2 r^2}$. This is the phase slip case.

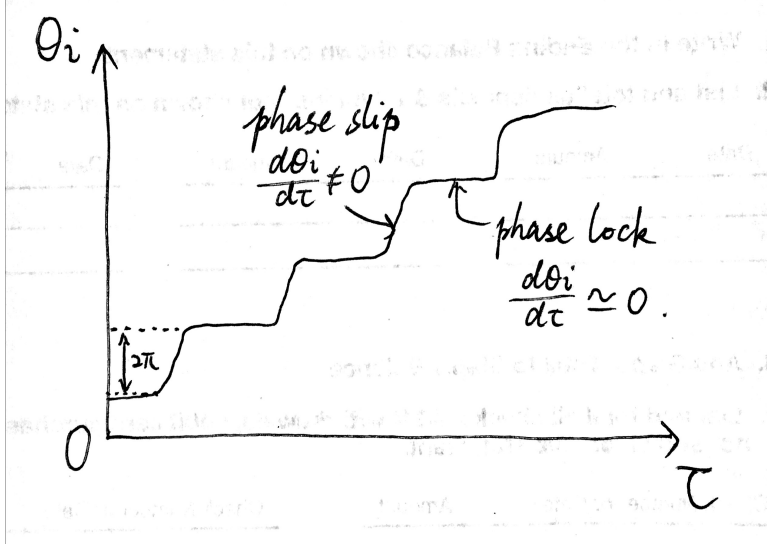


Figure 2: Phase lock and phase slip.

3.4 Phase dynamics in $N \rightarrow \infty$

The continuous system will be similar to the many oscillators system with the limit $N \rightarrow \infty$. The continuous limit should be phrased in terms of density. Define phase distribution $\rho(\theta, \tau, \omega)$, and it should be nonnegative, 2π -periodic in θ , and satisfy the normalization for all t and ω :

$$\int_{-\pi}^{\pi} \rho(\theta, \tau, \omega) d\theta = 1 \quad (55)$$

The evolution of ρ is governed by the continuity equation:

$$\frac{\partial}{\partial \tau} \rho + \frac{\partial}{\partial \theta} (\rho \dot{\theta}) = 0 \quad (56)$$

For steady state, we have $\frac{\partial}{\partial \tau} \rho = 0 \Rightarrow \rho \dot{\theta} = C$.

For phase locked oscillators, $\dot{\theta} = 0$, thus $\rho(\theta, \omega) = \frac{1}{2\pi} \delta(\theta - \frac{\omega}{Kr})$. This is the simplest case, which is called the uniform incoherent state.

For phase slip oscillators, $\rho(\theta, \omega) = \frac{C}{|\omega - Kr \sin \theta|}$ according to Eq. (54). $C = \frac{1}{2\pi} \sqrt{\omega^2 - (Kr)^2}$.

Now we focus on how the order parameter r change with K . Recall that in the last section, the expression for the order parameter r is Eq. (51)

$r = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}$ (Recall that $\Psi = 0$ by assumption). In the $N \rightarrow \infty$ limit, the expression now becomes:

$$r = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} e^{i\theta} \rho(\theta, \omega) g(\omega) d\theta d\omega \quad (57)$$

where $g(\omega)$ is the frequency distribution normalized by $\int_{-\infty}^{\infty} g(\omega) d\omega = 1$

Using angular brackets to denote population averages, we have

$$\langle e^{i\theta} \rangle = \langle e^{i\theta} \rangle_{lock} + \langle e^{i\theta} \rangle_{slip} \quad (58)$$

Note that $\langle e^{i\theta} \rangle = r e^{i\Psi} = r$. Thus

$$r = \langle e^{i\theta} \rangle_{lock} + \langle e^{i\theta} \rangle_{slip} \quad (59)$$

Phase locked case: because $g(\omega) = g(-\omega)$ and $\sin \theta_i = \frac{\omega_i}{Kr}$ for the phase locked case, we have $\langle \sin \theta \rangle_{lock} = 0$. So

$$\langle e^{i\theta} \rangle_{lock} = \langle \cos \theta \rangle_{lock} \quad (60)$$

$$= \int_{-Kr}^{Kr} \cos \theta(\omega) g(\omega) d\omega \quad (61)$$

$$= Kr \int_{-\pi/2}^{\pi/2} \cos^2 \theta g(Kr \sin \theta) d\theta \quad (62)$$

where $\theta(\omega)$ is defined by $\sin \theta_i = \frac{\omega_i}{Kr}$.

Phase slip case:

$$\langle e^{i\theta} \rangle_{slip} = \int_{-\pi}^{\pi} \int_{|\omega| > Kr} e^{i\theta} \rho(\theta, \omega) g(\omega) d\theta d\omega \quad (63)$$

because of the symmetry $\rho(\theta + \pi, -\omega) = \rho(\theta, \omega)$ and $g(\omega) = g(-\omega)$, the above integral vanishes.

So in summary, we obtain the expression for the order parameter r :

$$r = Kr \int_{-\pi/2}^{\pi/2} \cos^2 \theta g(Kr \sin \theta) d\theta \quad (64)$$

Let's examine this function as $r = r(K)$. This function has a trivial solution $r = 0$ for any K . This is corresponding to the totally incoherent case. Another solution satisfies:

$$1 = K \int_{-\pi/2}^{\pi/2} \cos^2 \theta g(Kr \sin \theta) d\theta \quad (65)$$

This branch bifurcates continuously from $r = 0$ at a value $K = K_c$ obtained by letting $r \rightarrow 0^+$ in the above equation. Thus

$$\boxed{K_c = 2/\pi g(0)} \quad (66)$$

where $g(0)$ is a given function at a given value so it is known. This is Kuramotos exact formula for the critical coupling at the onset of collective synchronization.

For $r < 1$, we can do the Taylor expansion:

$$g(Kr) = g(0) + g'(0)Kr \sin \theta + \frac{1}{2}g''(0)(Kr \sin \theta)^2 + \dots \quad (67)$$

Substituting it yields:

$$r \simeq \sqrt{\frac{16}{\pi K_c^3}} \sqrt{\frac{\mu}{-g''(0)}} \quad (68)$$

where $\mu = \frac{K-K_c}{K_c}$.

Near the onset, we have the scaling law:

$$\frac{\partial r}{\partial K} \sim \frac{1}{\sqrt{K - K_c}} \quad (69)$$

The bifurcation is supercritical if $g''(0) < 0$ and it is subcritical if $g''(0) > 0$.

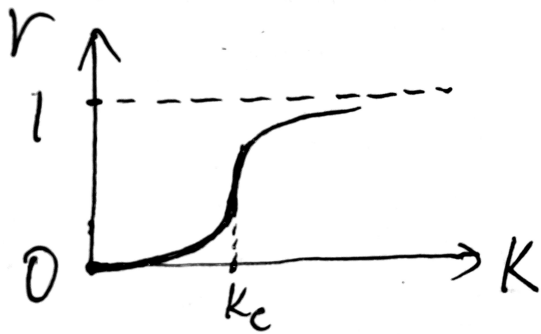


Figure 3: r as a function of K .

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